Finite Element Modeling of Scattering from Underwater Buried and Proud Munitions

MR-2408
Dr. Ahmad T. Abawi
HLS Research
In-Progress Review Meeting
May 17, 2018

MR-2408: Project Title

Performer: Ahmad T. Abawi

Technology Focus

- Development and validation of computational tools to aid detection and identification of military munitions (UXOs) found in pond, lakes, rivers, estuaries and coastal ocean areas

Research Objectives

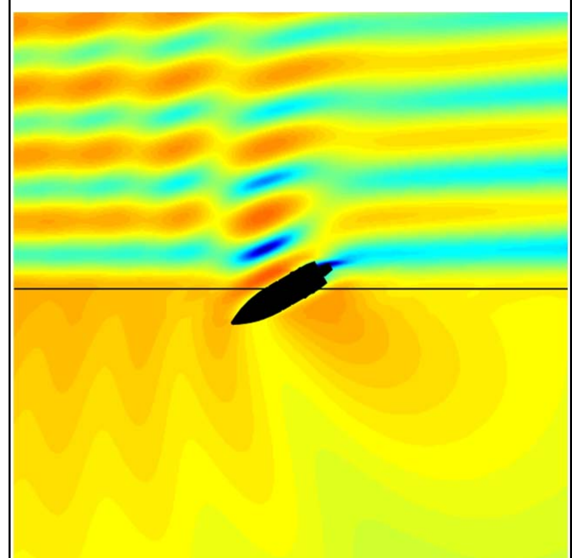
- *Develop finite element computational tools to model scattering of sound from UXOs found in shallow water regions*
- *Criteria for success are based on how well modeled results agree with measurements*

Project Progress and Results

- *Was able to speed up previously developed and validated scattering models for scattering by UXOs in a shallow water environment by as much as a factor of 10*

Technology Transition

- *This technology has direct application in MCM and ASW. It also has applications in industry ranging from modeling noise generated from wind turbines to detecting and localizing oil reservoirs*



Project Team

- Ahmad T. Abawi (HLS Research)
- Professor Petr Krysl (Structural Engineering Dept., University of California, San Diego)

Technical Objectives

- In this phase of the project, our technical objective is to substantially reduce the execution times of, previously-developed and validated, numerical models

- Use these models to
  - Identify and understand robust features in the recorded echoes of UXOs to facilitate their classification
  - Analyze and interpret experimental data
  - Produce a library of acoustic color for various targets
  - Plan future experiments

- *Codes that run faster, achieve these goals much better*

Technical Approach

Background

To compute the acoustic color, we compute the backscattered acoustic field from a sonar as a function of aspect angle and frequency

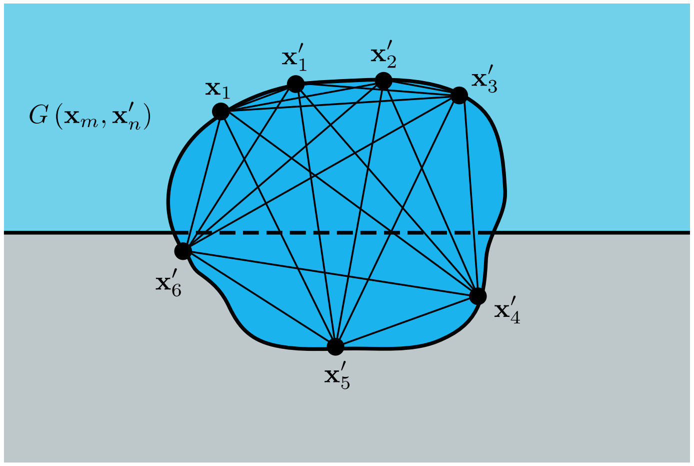
$$P(f,\theta) = \left(\mathbf{A} - i\omega\mathbf{B}\mathbf{L}\left(-\omega^2\mathbf{M}+\mathbf{K}\right)^{-1}\mathbf{L}^T\right)^{-1} P_{inc}(f,\theta),$$

Where $A_{ij} = \frac{\delta_{ij}}{2} - \int_{s_j} \frac{\partial G(x_i, x_j')}{\partial n_j'} ds_j'$, $B_{ij} = -i\omega\rho \int_{s_j} G(x_i, x_j') ds_j'$

If there are n surface elements, computation of A and B require n^2 function evaluations

For a typical problem this is on the order of 100,000,000 function evaluations

We would like to reduce this number without compromising accuracy

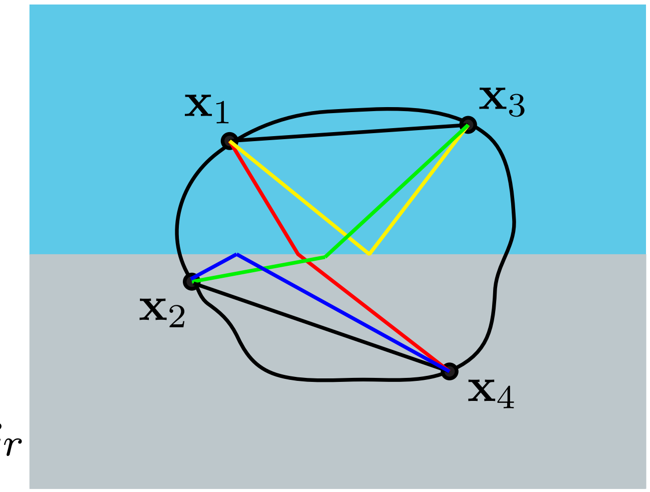


Technical Approach

Background

$$G(0,z_1;r_3,z_3) = \int_0^\infty \frac{i\mathbb{R}e^{ik_{z_1}(z_1+z_3)}}{4\pi k_{z_1}} J_0(k_r|r_3|)k_r dk_r$$

$$+\text{Direct Field},$$



$$G(0,z_2;r_4,z_4) = \int_0^\infty \frac{-i\mathbb{R}e^{ik_{z_2}(z_2+z_4)}}{4\pi k_{z_2}} J_0(k_r|r_4|)k_r dk_r$$

$$+\text{Direct Field}, \quad \sqrt{k_r^2+kz_1^2}=\omega/c_1, \quad \sqrt{k_r^2+kz_2^2}=\omega/c_2.$$

$$G(0,z_1;r_4,z_4) = \int_0^\infty \frac{i\mathbb{T}_{12}e^{ik_{z_1}z_1}e^{-ik_{z_2}z_4}}{4\pi k_{z_1}} J_0(k_r|r_4|)k_r dk_r,$$

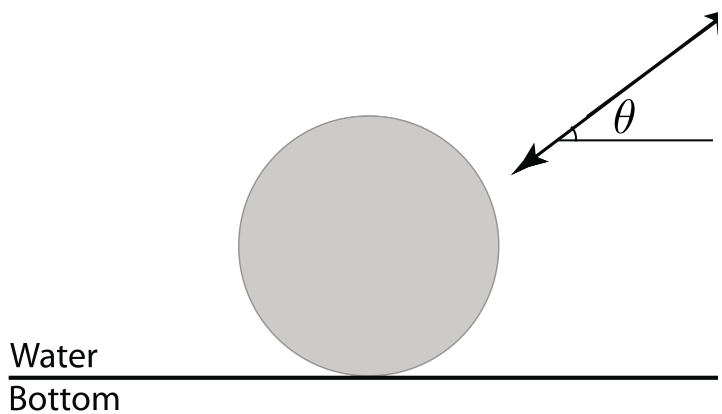
All require n^2 evaluations!

$$G(0,z_2;r_3,z_3) = \int_0^\infty \frac{i\mathbb{T}_{21}e^{-ik_{z_2}z_2}e^{ik_{z_2}z_3}}{4\pi k_{z_2}} J_0(k_r|r_3|)k_r dk_r.$$

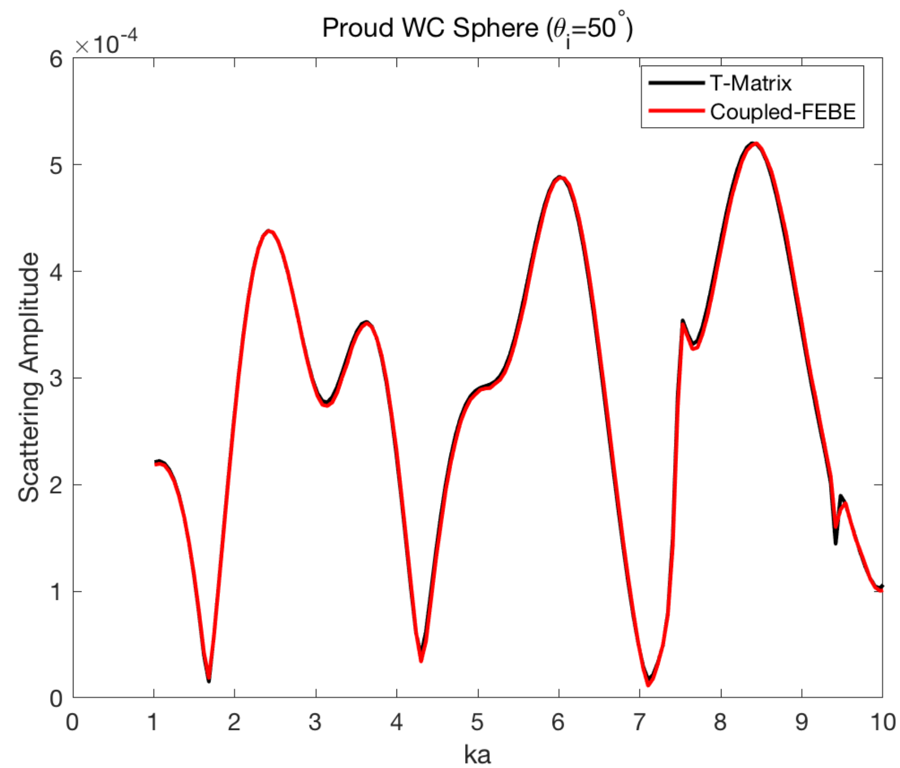
Technical Approach

Validation

Proud WC sphere radius = 6.35 mm



$$c_w = 1485\ m/s,\quad \rho_w = 1000\ kg/m^3$$
$$c_b = 1655\ m/s,\quad \rho_b = 1890\ kg/m^3.$$

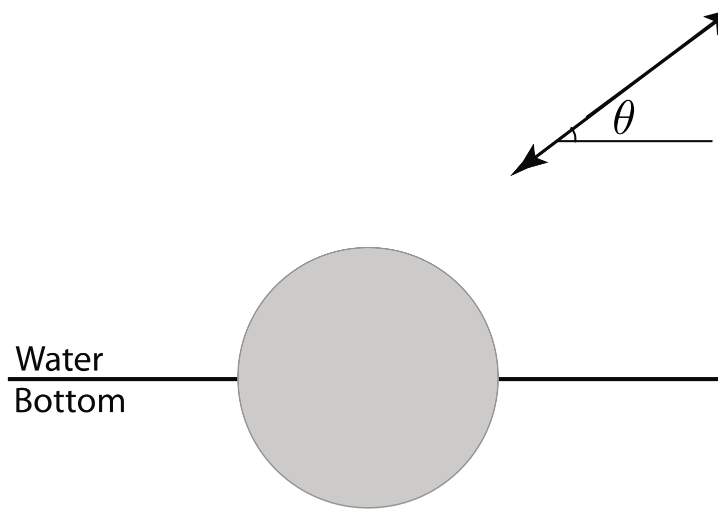


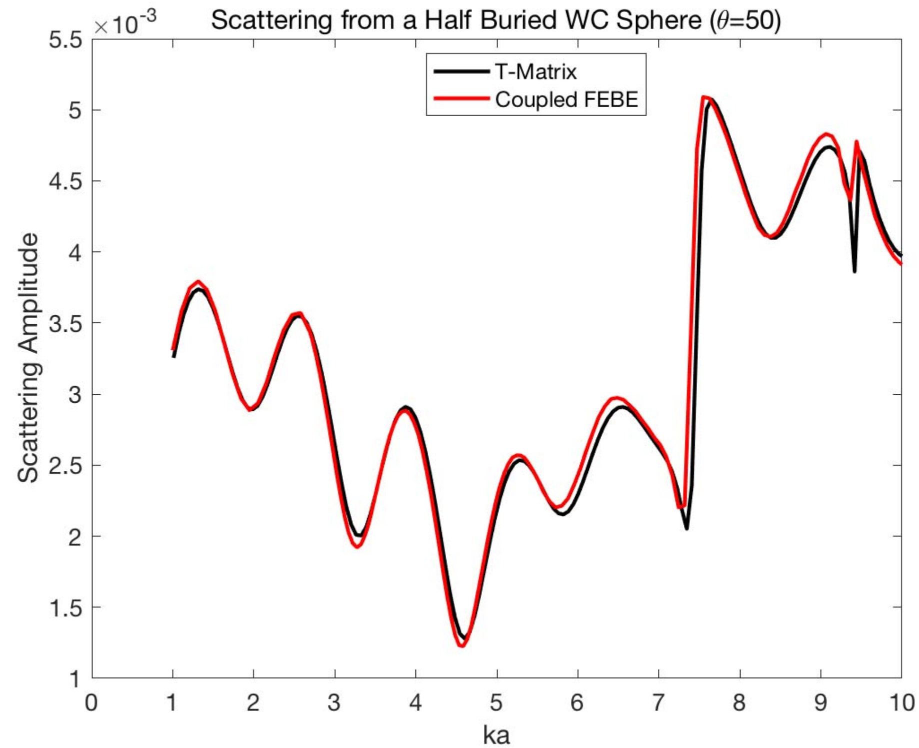
Raymond Lim, "Acoustic scattering by a partially buried three-dimensional elastic obstacle", JASA (104) , 769 (1998)

Technical Approach

Validation

Half buried WC sphere radius = 6.35 mm

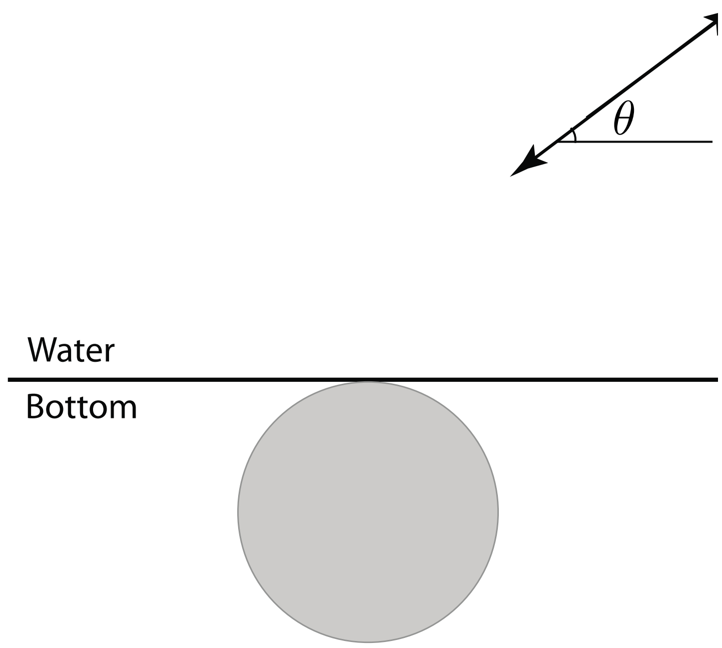


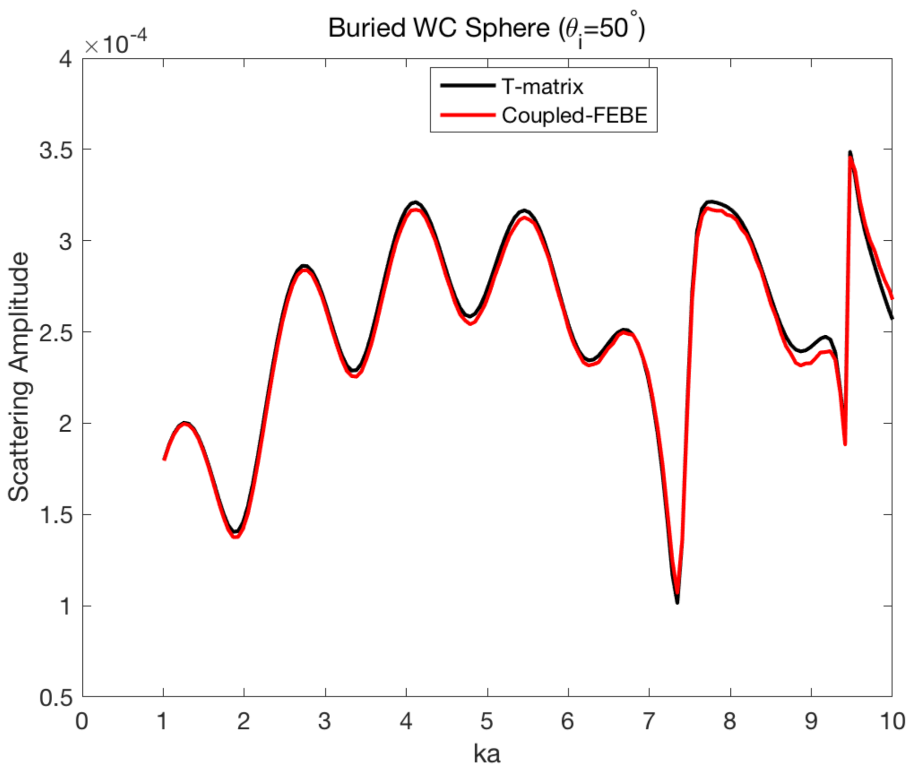


Technical Approach

Validation

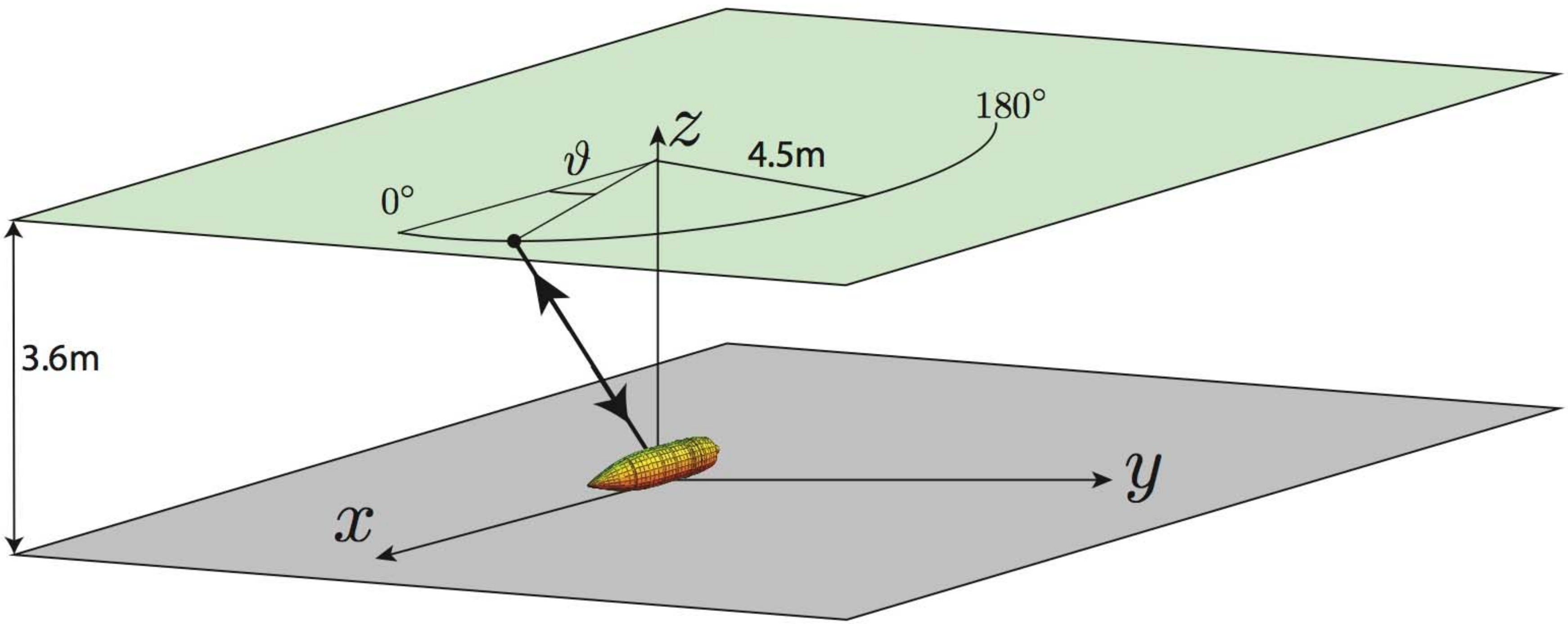
Buried WC sphere radius = 6.35 mm





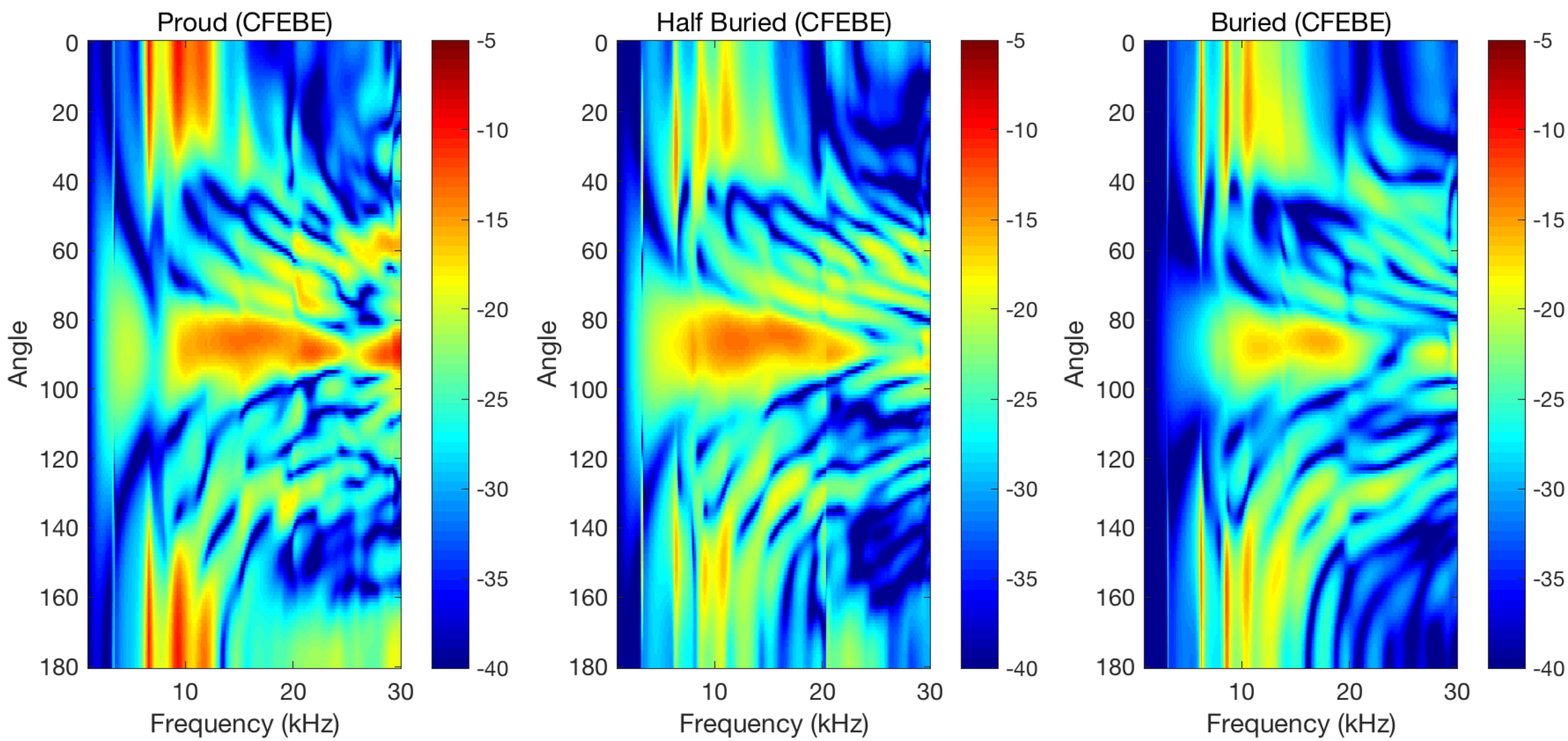
Results

Aluminum UXO replica



Technical Approach

Results



48 hours

150 hours

48 hours

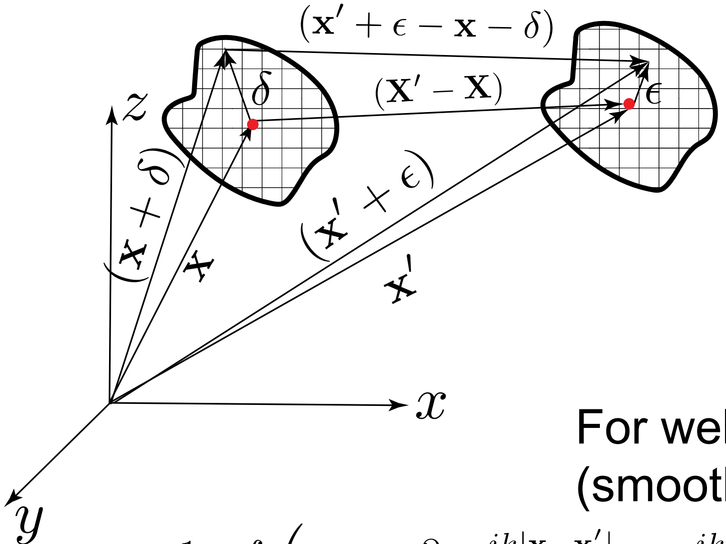
Technical Approach

Hierarchical Matrices

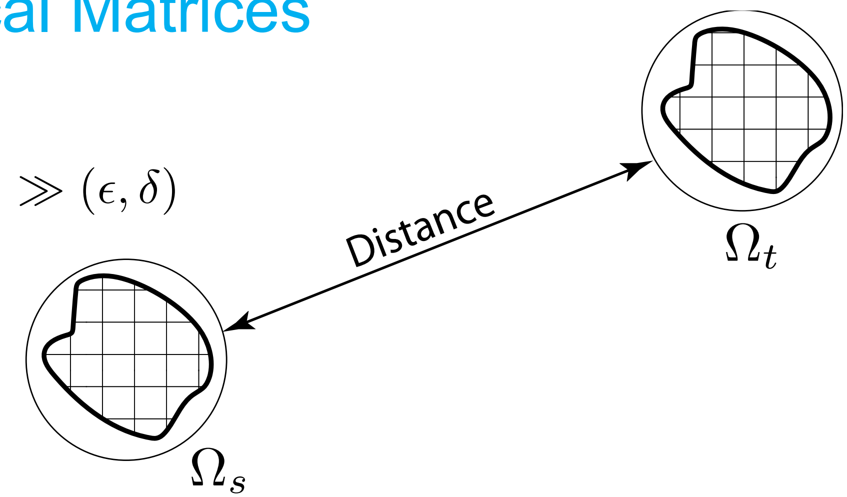
- Boundary element formulation produces large dense matrices, whose assembly and processing take an enormous amount of time
- Look for ways to reduce numerical complexity, particularly for methods that reduce matrix assembly time
- The fast multipole and hierarchical matrices methods reduce typical matrix operations from $\mathscr{O}(N^2)$ to $\mathscr{O}(N \log N)$
- The latter also reduces matrix assembly time by the same amount

Hierarchical Matrices

Low-rank approximation of the kernel



$$|\mathbf{x}' - \mathbf{x}| \gg (\epsilon, \delta)$$



For well-separated clusters the kernel is slowly-varying (smooth), allowing low-rank approximation

$$p(\mathbf{x}) = \frac{1}{4\pi}\int \left(p(\mathbf{x}')\frac{\partial}{\partial n'}\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} - \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\frac{\partial}{\partial n'}p(\mathbf{x}')\right) dS',\ \mathbf{X}\in\mathbb{R}^n,\ \mathbf{X}'\in\mathbb{R}^m\ \text{(full rank)},$$

$$p(\mathbf{x}) = \frac{1}{4\pi}\int \left(p(\mathbf{x}')\frac{\partial}{\partial n'}\frac{e^{ik|\mathbf{x}+\epsilon-\mathbf{x}'-\delta|}}{|\mathbf{x}+\epsilon-\mathbf{x}'-\delta|} - \frac{e^{ik|\mathbf{x}+\epsilon-\mathbf{x}'-\delta|}}{|\mathbf{x}+\epsilon-\mathbf{x}'-\delta|}\frac{\partial}{\partial n'}p(\mathbf{x}')\right) dS',$$

$$|\mathbf{x}'+\epsilon-\mathbf{x}+\delta| = |\mathbf{x}'-\mathbf{x}| + \hat{r}\cdot\epsilon - \hat{r}\cdot\delta + \mathcal{O}\left(\frac{\epsilon,\delta}{|\mathbf{x}'-\mathbf{x}|^2}\right), \qquad r \equiv |\mathbf{x}'-\mathbf{x}|,\ \hat{r} = \frac{(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|},\ \mathbf{k} = k\hat{r},$$

$$p(\mathbf{x}) = \frac{e^{ikr}}{4\pi r}\int \left(p(\mathbf{x}')\frac{\partial}{\partial n'}e^{-i\mathbf{k}\cdot(\epsilon-\delta)} - e^{-i\mathbf{k}\cdot(\epsilon-\delta)}\frac{\partial}{\partial n'}p(\mathbf{x}')\right) dS', (r,\mathbf{x}')\in\mathbb{R}^1,\ \delta\in\mathbb{R}^n,\ \epsilon\in\mathbb{R}^m,$$

which can be expressed as a rank 1 outer product

$$p(\mathbf{x}) = AB^T.$$

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \cdot \\ \cdot \\ \delta_n \end{pmatrix} \begin{pmatrix} \epsilon_1 & \epsilon_2 & \cdot & \cdot & \epsilon_m \end{pmatrix}$$

Technical Approach

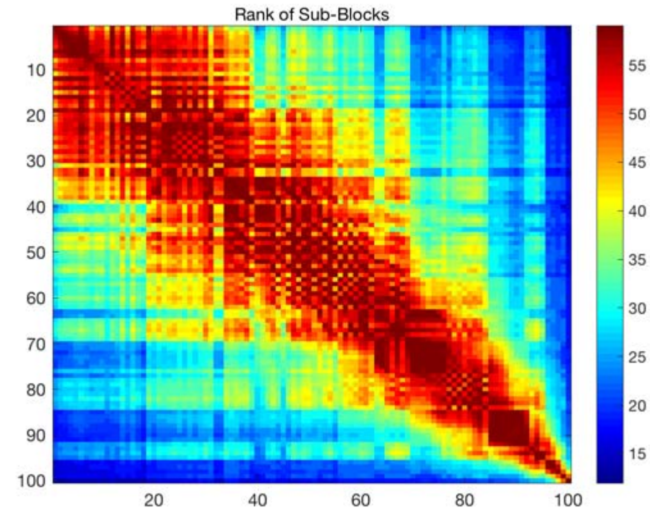
Hierarchical Matrices

Any matrix $M \in \mathbb{R}^{m\times n}$ $(m \leq n)$ can be represented as an outer product (SVD)

$$M = \sum_{i=1}^{k} \lambda_i u_i v_i^T,$$

Where λ are the singular values, u and v are orthonormal matrices, and $k \leq \min(m,n)$

A matrix is low rank when $k \ll \min(m,n)$

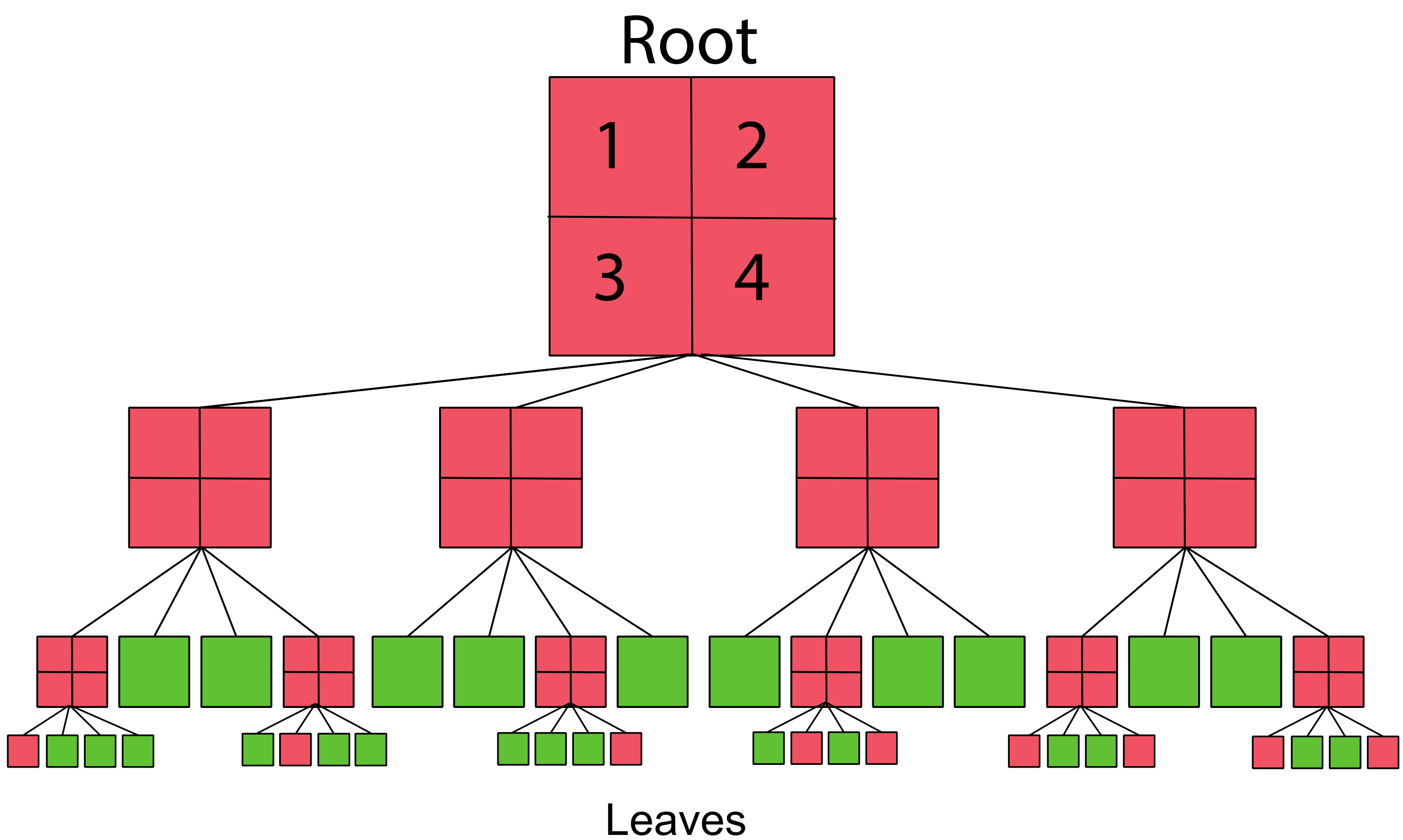


Low-rank matrices written in the form of an outer product, require only $k \times (m+n)$ operations to fill-in, instead of $m \times n$ operations for full-rank matrices.

They also require only $2k \times (m+n) - k$ FLOPS for matrix-vector multiplication, as opposed to $m \times n$ FLOPS for full-rank matrices

Hierarchical matrices provides a systematic and mathematically rigorous way of finding low-rank sub-blocks in a matrix based on a predefined error tolerance level

Matrix Partitioning

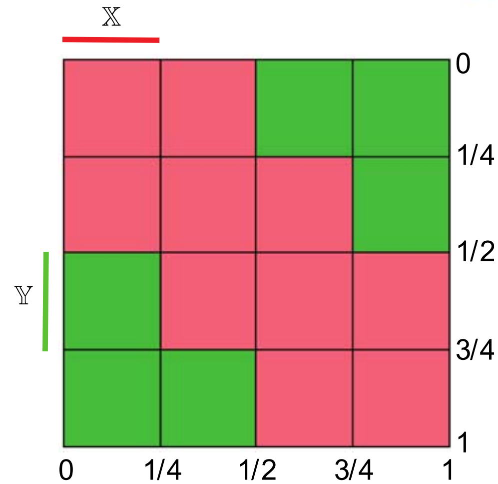


Starting from the root (the entire matrix), subdivide each block until either the admissibility condition is satisfied or the block is sufficiently small that it cannot be subdivided. If the admissibility condition is satisfied, the sub-block would be a low-rank block (green), otherwise it would be full-rank (red).

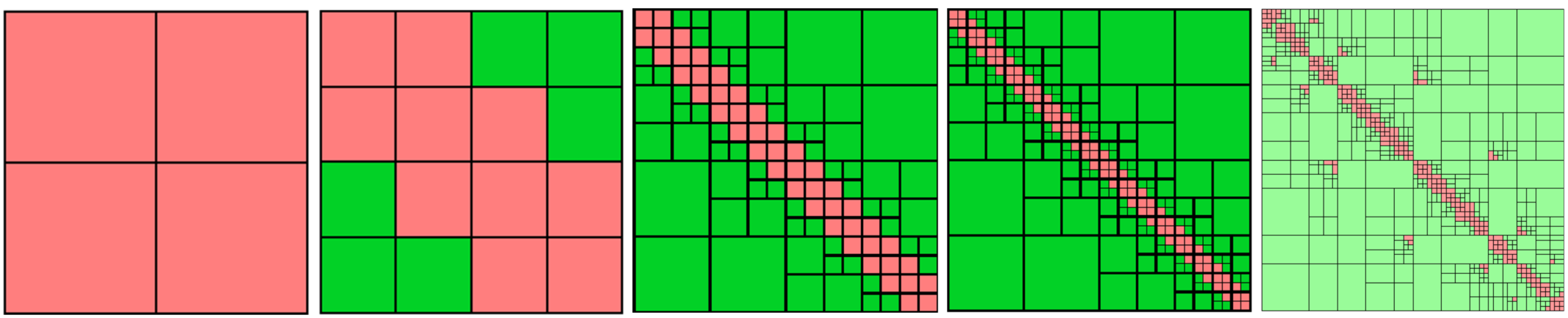
Admissibility condition:

$$\max\{\mathrm{diam}(\mathbb{X}), \mathrm{diam}(\mathbb{Y})\} \leq \eta \mathrm{dist}(\mathbb{X}, \mathbb{Y})$$

Simply determines how close the sub-block in question is to the diagonal, where η is an adjustable parameter



Continuing the process results in hierarchical matrices, reducing both matrix operations and assembly from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log N)$, where N is the original size of the matrix.



To be able to fill the green blocks, we need a low-rank representation of half-space Green's functions

Technical Approach

The multipole solution in two half-spaces

To be able to use H-Matrices, we need to be able to express our Green's functions in the form of an outer product. Our current solution

$$G(r_s, z_s; r_r, z_r) = \int_0^\infty \frac{i\mathbb{R}e^{ik_{z_1}(z_s+z_r)}}{4\pi k_{z_1}} J_0(k_r|r_{sr}|)k_r dk_r,$$

can not be expressed as an outer product. This solution comes from the propagation community and it is a 2D solution of Helmholtz equation in cylindrical coordinates, with the source and receiver in the same radial plane

We need the same solution in 3D for a source located at (r_s, z_s, φ_s)

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + k^2\right] \psi(r, z, \varphi) = S_w \frac{\delta(r - r_s)\delta(z - z_s)\delta(\varphi - \varphi_s)}{2\pi r}$$

Application of an azimuthal Fourier series

$$\psi(r, z, \varphi) = \sum_{n=-\infty}^{\infty} \psi_n(r, z) e^{in\varphi},$$

Technical Approach

The multipole solution in two half-spaces

followed by a Hankel transform

$$\psi_n(r,z) = \int_0^\infty \psi_n(k_r,z) J_n(k_r r) k_r dk_r,$$

to get

$$\left[\frac{d^2}{dz^2} + \left(k^2 - k_r^2\right)\right]\psi_n(k_r,z) = S_w \frac{\delta(z-z_s)}{2\pi} e^{-in\varphi_s} J_n(k_r r_s),$$

which has a solution

$$\psi(r_r, z_r, \varphi_r; r_s, z_s, \varphi_s) = \frac{iS_w}{4\pi}\sum_{n=0}^{\infty}\epsilon_n \int_0^\infty \frac{e^{ik_z|z_r - z_s|}}{k_z} J_n(k_r r_r) J_n(k_r r_s)\cos[n(\varphi_r - \varphi_s)] k_r dk_r$$

This is the multipole representation of the free space Green's function in cylindrical coordinates, which can be expressed as an outer product since

$$\cos[n(\varphi_s - \varphi_r)] = \cos(n\varphi_s)\cos(n\varphi_r) + \sin(n\varphi_s)\sin(n\varphi_r)$$

Small n produces a low-rank solution

Technical Approach

The multipole solution in two half-spaces

The old solution can be recovered from this, more general, solution by setting $r_s = 0$,

$$\psi\left(r_r, z_r; 0, z_s\right) = \frac{iS_w}{4\pi}\int_0^\infty \frac{e^{ik_z|z_r - z_s|}}{k_z} J_0(k_r r_r) k_r dk_r$$

The solution for two half-spaces can be obtained for each azimuthal order by scaling the source term by $e^{-in\varphi_s} J_n(k_r r_s)$, which gives

$$G(r, z, \varphi; r_s, z_s, \varphi_s) = \sum_{n=0}^{\infty} \epsilon_n \int_0^\infty \frac{i\mathbb{R}e^{ik_{z_1} z_s} e^{ik_{z_1} z_r}}{4\pi k_{z_1}} J_n(k_r r_s) J_n(k_r r_r) \cos\left[n(\varphi_s - \varphi_r)\right] k_r dk_r$$
$$+\text{Direct Field}, \; (z, \; z_s > 0),$$

$$G(r, z, \varphi; r_s, z_s, \varphi_s) = \sum_{n=0}^{\infty} \epsilon_n \int_0^\infty \frac{-i\mathbb{R}e^{ik_{z_2} z_s} e^{ik_{z_2} z_r}}{4\pi k_{z_2}} J_n(k_r r_s) J_n(k_r r_r) \cos\left[n(\varphi_s - \varphi_r)\right] k_r dk_r$$
$$+\text{Direct Field}, \; (z, \; z_s < 0).$$

Technical Approach

The multipole solution in two half-spaces

For the transmitted field from water to bottom and from bottom to water, the solutions are respectively given by

$$G(r,z,\varphi;r_s,z_s,\varphi_s)=\sum_{n=0}^{\infty}\epsilon_n\begin{cases}\displaystyle\int_0^\infty \frac{i\mathbb{T}_{12}e^{ik_{z_1}z_s}e^{-ik_{z_2}z}}{4\pi k_{z_1}}J_n\left(k_r r_s\right)J_n\left(k_r r_r\right))\cos\left[n(\varphi_s-\varphi_r)\right]k_r dk_r,\, z_s\geq\ z<0,\\ \\ \displaystyle\int_0^\infty \frac{i\mathbb{T}_{21}e^{ik_{z_1}z}e^{-ik_{z_2}z_s}}{4\pi k_{z_2}}J_n\left(k_r r_s\right)J_n\left(k_r r_r\right))\cos\left[n(\varphi_s-\varphi_r)\right]k_r dk_r,\, z_s<0,\ z\geq 0.\end{cases}$$

Where in the above equations $\sqrt{k_r^2+kz_1^2}=\omega/c_1,\ \sqrt{k_r^2+kz_2^2}=\omega/c_2,$

$$\mathbb{R}=\frac{k_{z_1}\rho_2-k_{z_2}\rho_1}{k_{z_1}\rho_2+k_{z_2}\rho_1},\quad \mathbb{T}_{12}=\frac{2k_{z_1}\rho_2}{k_{z_1}\rho_2+k_{z_2}\rho_1},\quad \mathbb{T}_{21}=\frac{2k_{z_2}\rho_1}{k_{z_1}\rho_2+k_{z_2}\rho_1}.$$

All these solution can be expressed as an outer product

Technical Approach

The multipole solution in two half-spaces

For m surface elements, these solutions only require $n \times m$ function evaluations instead of m^2 function evaluations.

Since $n \ll m$, this is a significant reduction

For a typical problem m=10000 and n is at most 30

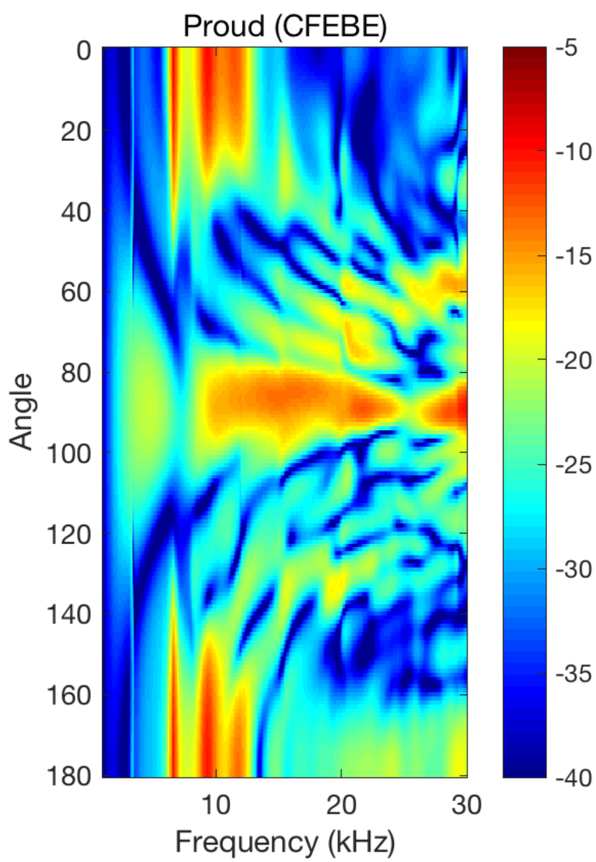
While our previous method required 100,000,000 function evaluations per frequency, the new method at most requires 300,000, fewer by a factor of at least 170!

We now have all the machinery needed to use H-Matrices to gain even more substantial computational speed

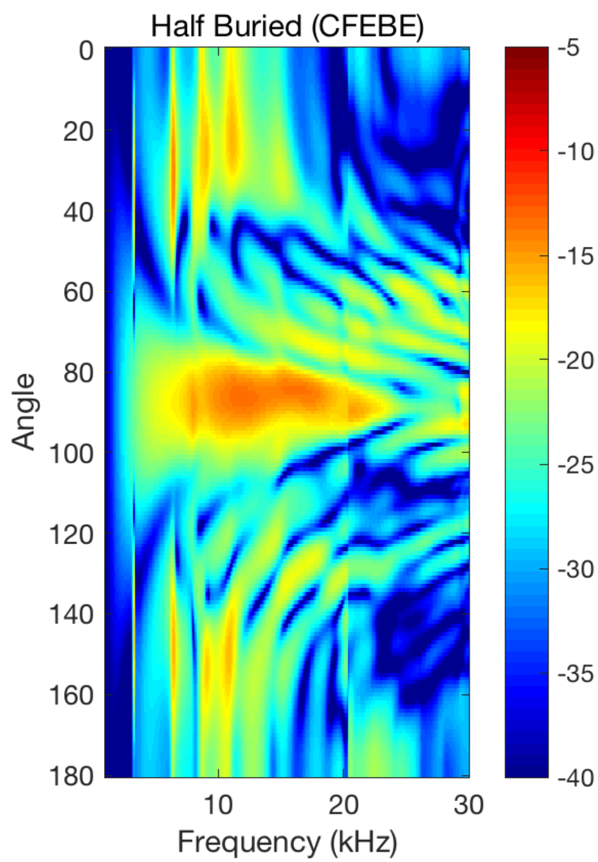
Even without H-Matrices, we see a significant improvement in computational speed by just using the new Green's function formulation, particularly in the case of partially buried targets

Results

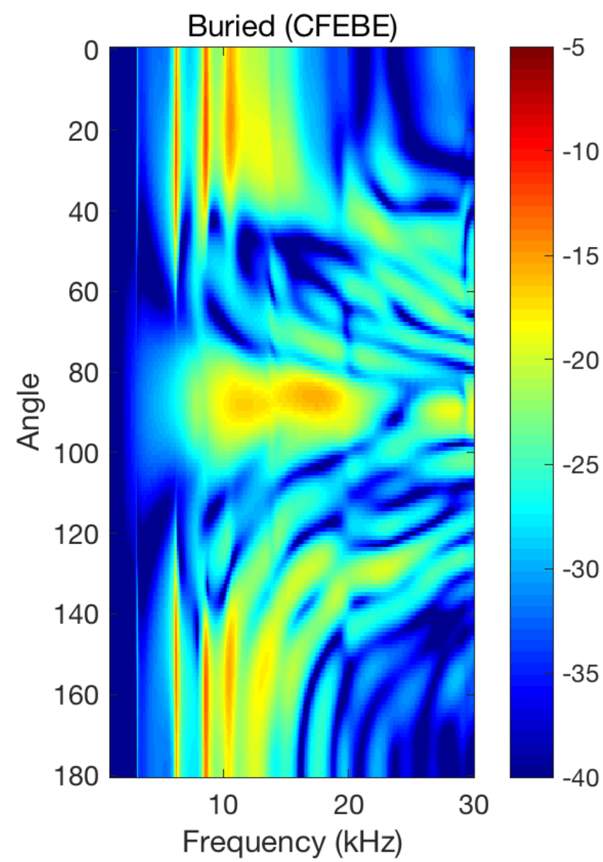
Old Results



48 hours



150 hours



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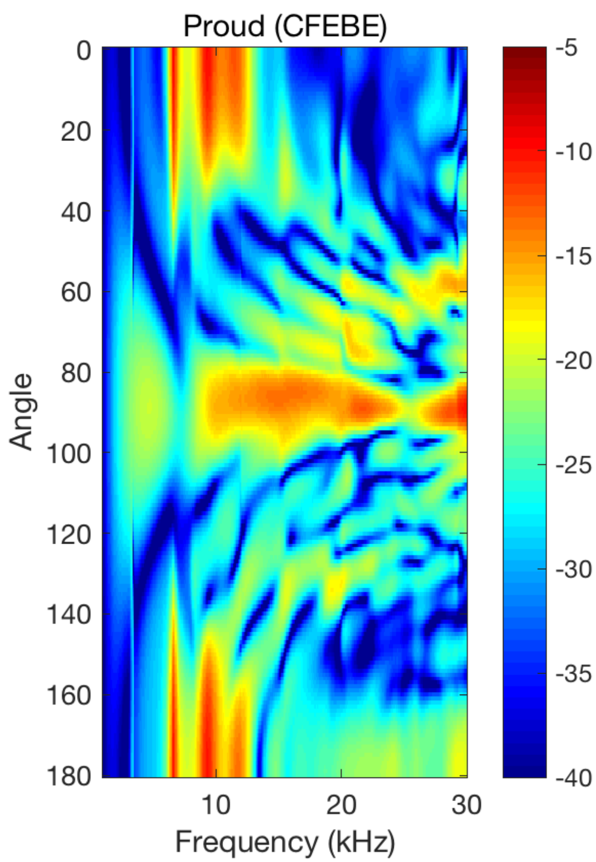
Results

New Results



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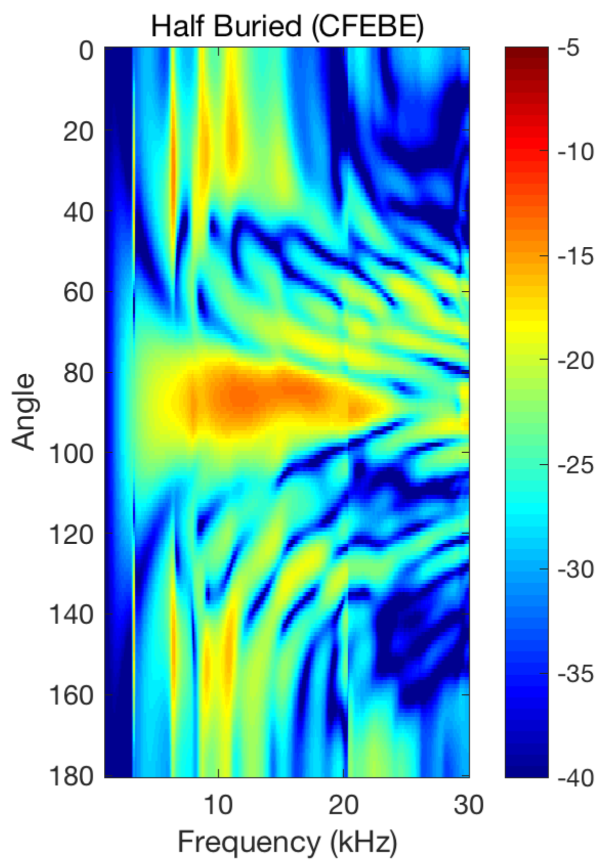

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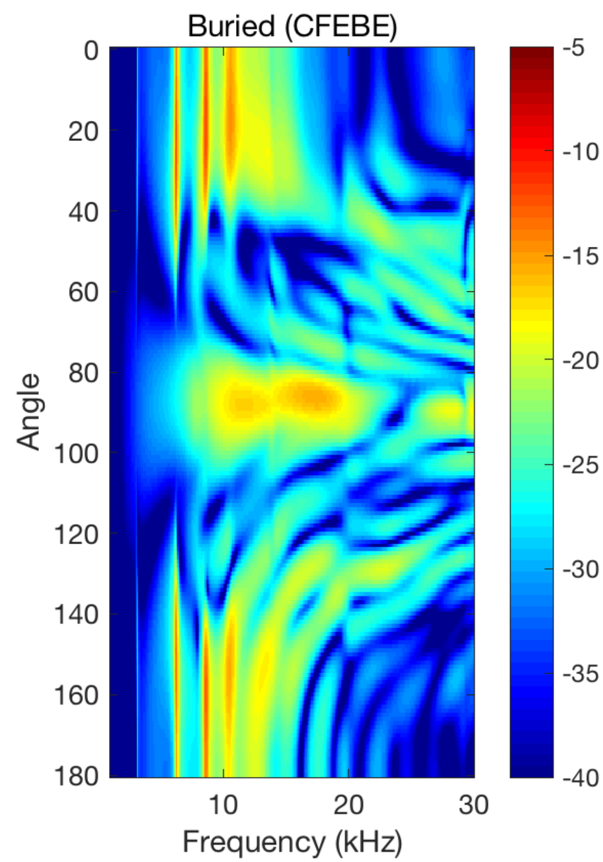
18 hours



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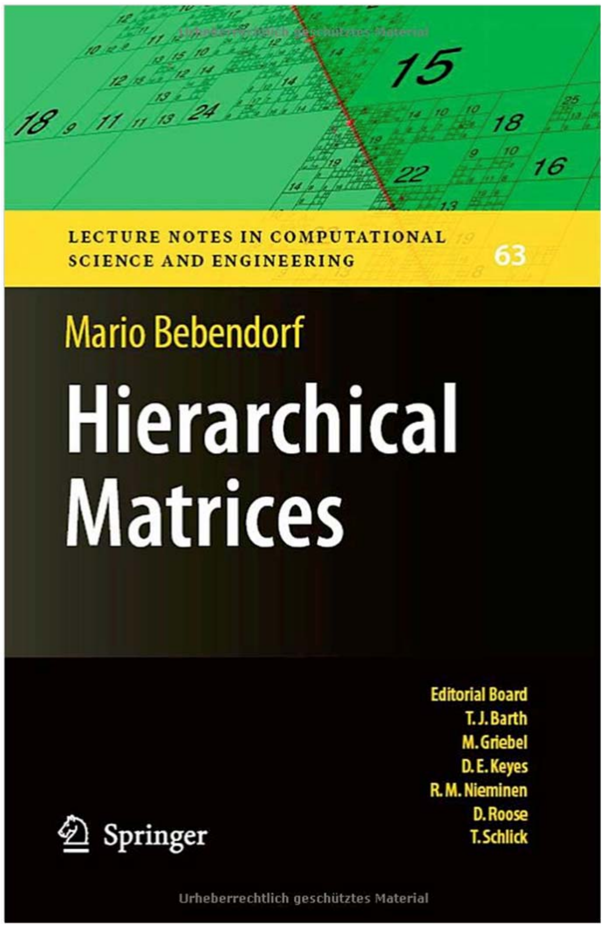
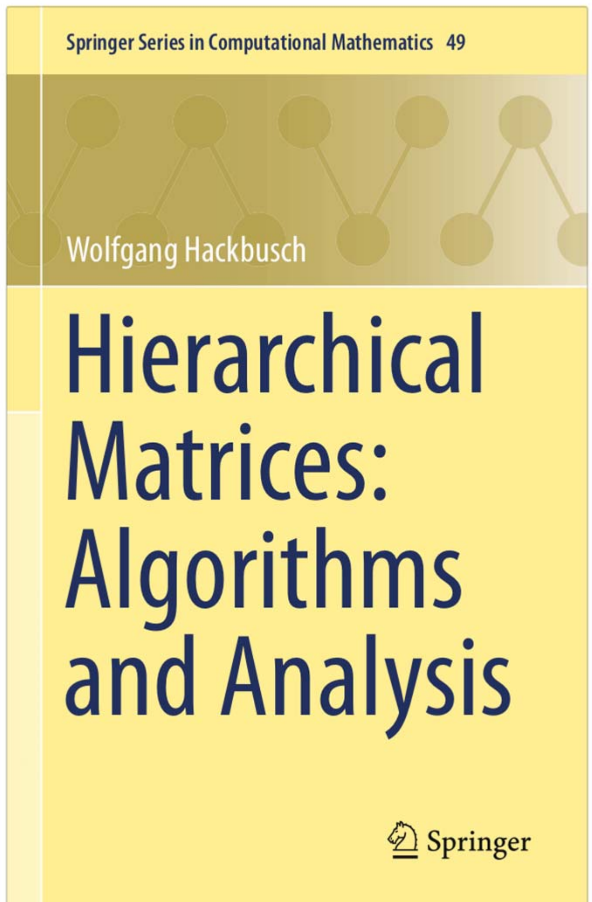
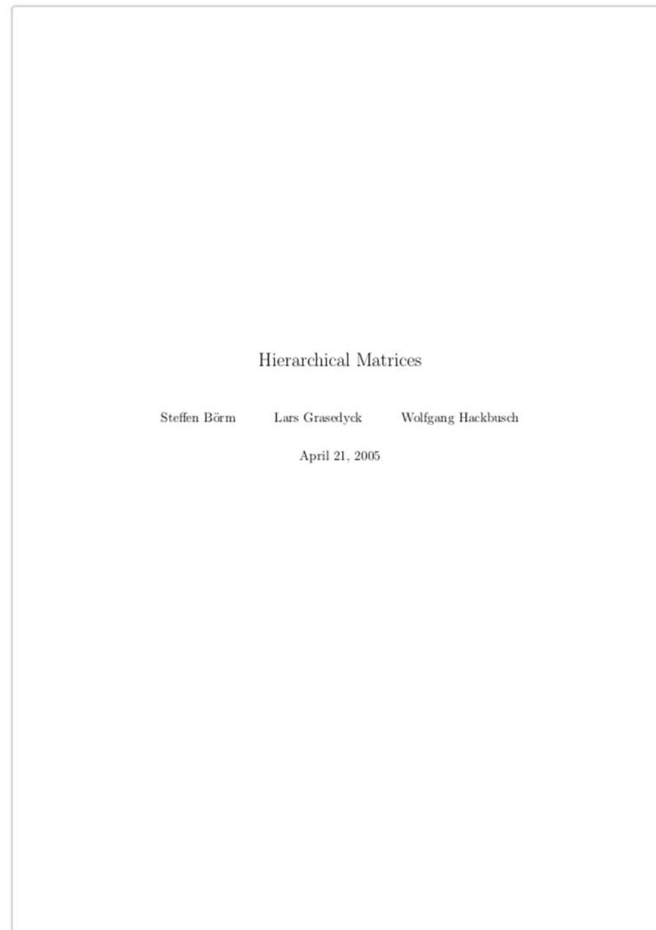
Technical Approach

Hierarchical Matrices







Technical Approach

Hierarchical Matrices

H2Lib 3.0

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H2Lib Documentation

The `H2Lib` package contains algorithms and data structures for working with hierarchical matrices [[5]], [[3]], [[6]] and $\mathcal{H}^2$-matrices [[4]], [[1]], [[2]]. It is being developed in the Scientific Computing Group of Kiel University, since we require a software library that can be used both for teaching and research purposes and the existing libraries currently do not meet both requirements.

In order to offer a good basis for teaching courses on hierarchical matrices, the modules have been organized in a layered design that allows students to work with the lower layers (e.g., for handling matrices and vectors) without having to worry about higher layers (e.g., approximative algebraic routines or sophisticated compression algorithms).

A researcher using `H2Lib` finds a relatively complete set of functions for creating and manipulating $\mathcal{H}$- and $\mathcal{H}^2$-matrices, e.g., for performing matrix-vector multiplications, approximative algebraic operations like multiplication, inversion and factorization, and functions for compressing and converting matrices between different representations.

For the sake of convenience, we have also included modules for a number of typical model problems, e.g., for boundary integral equations or elliptic partial differential equations, with the corresponding auxiliary modules for singular quadrature and simple grid management.

Technical Approach

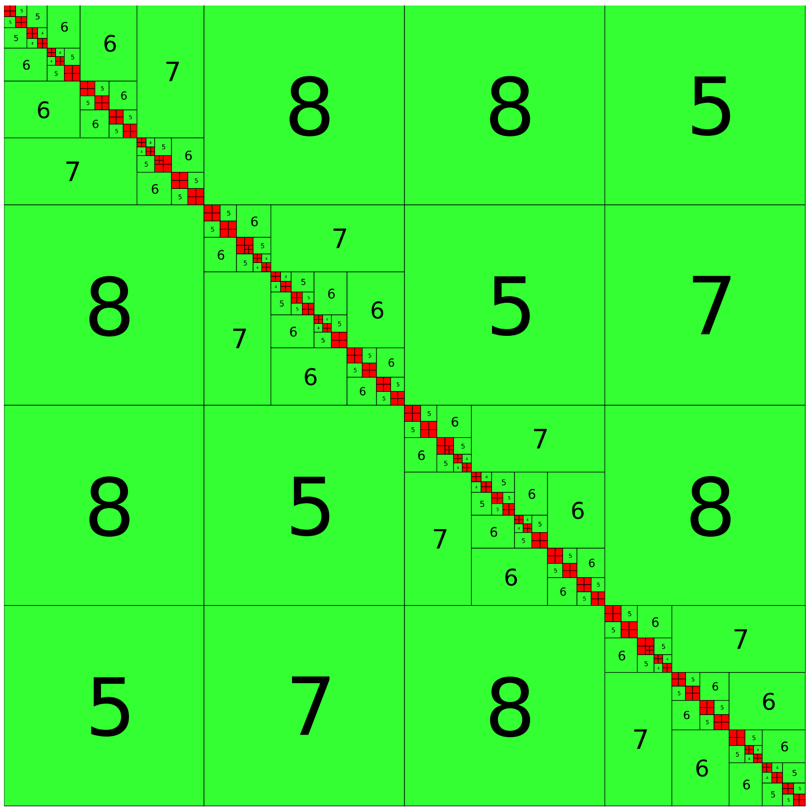
Hierarchical Matrices

- We plan to use the H2Lib to partition matrices and carry out the subsequent matrix operations inside the H2Lib codes since that would provide the most direct way of making use of the numerical advantages that the method offers
- For this purpose, we have decided to use Julia instead of Matlab to interact with this library, as Julia allows calling C routines directly and there would be no need for Mex programming
- As an added advantage, our codes run more than twice as fast in Julia compared to Matlab
- Converting Matlab to Julia is straight forward and we have already converted our main routines

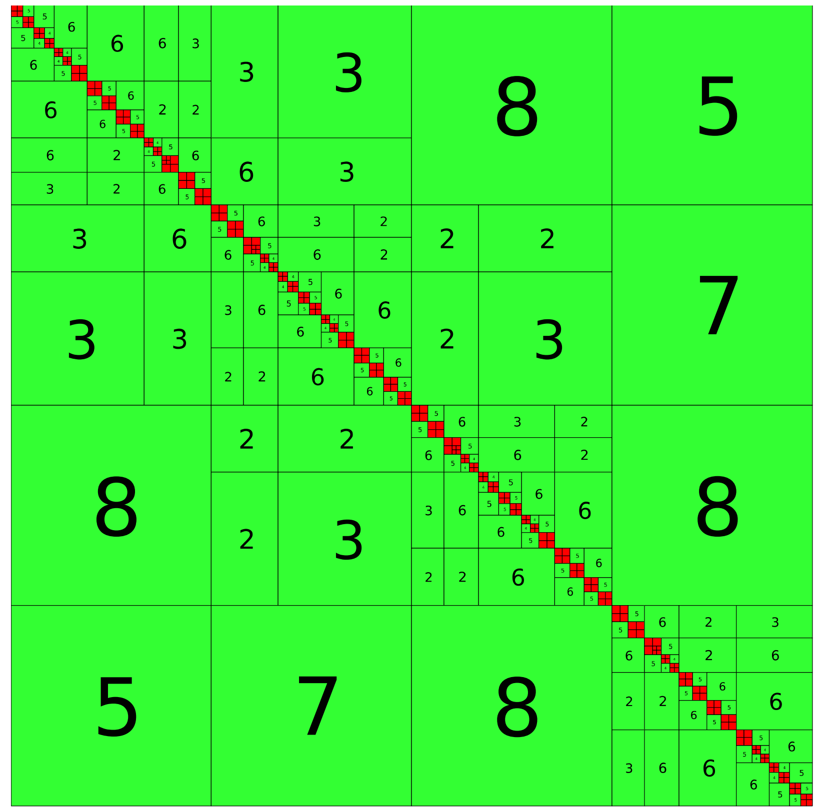
Results

Solution of Laplace Equation using Hierarchical Matrices

$\eta = 1$



$\eta = 0.5$

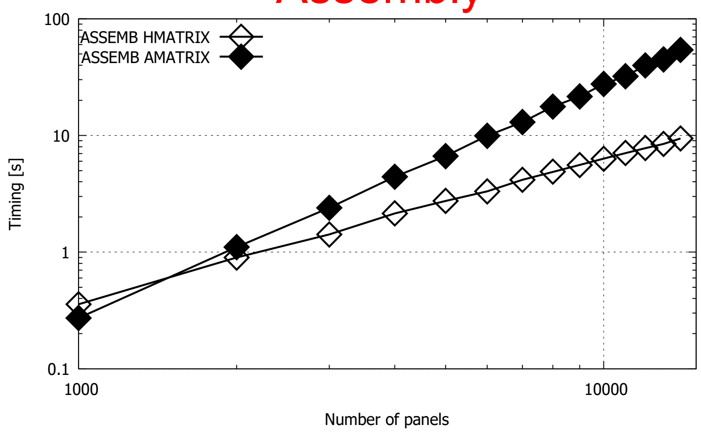


Results

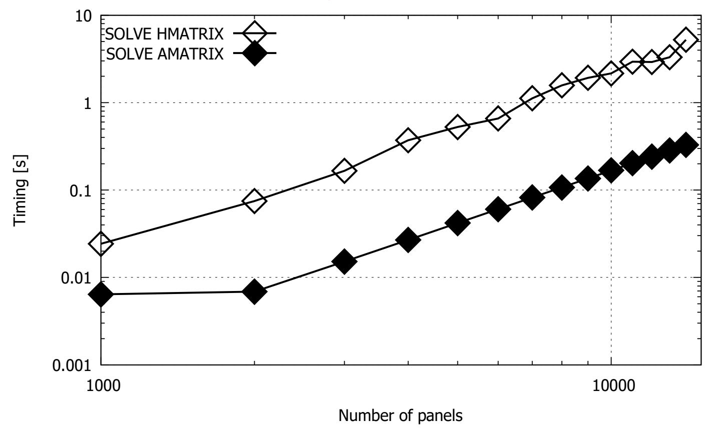
Solution of Laplace Equation using Hierarchical Matrices

Conjugate Gradient Method

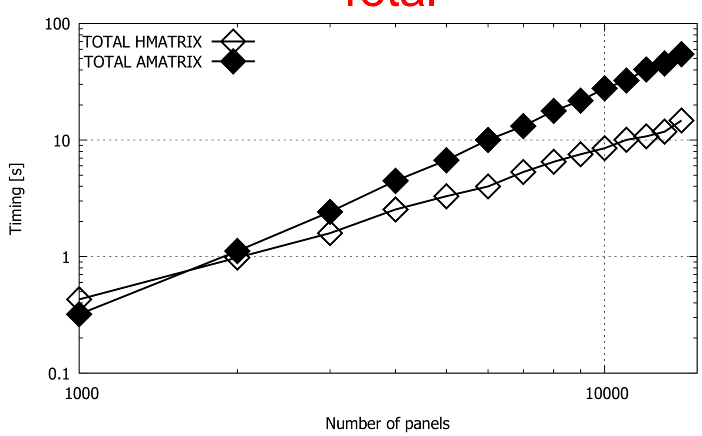
Assembly



Solve



Total



$$\text{Total Time} \sim n^2$$

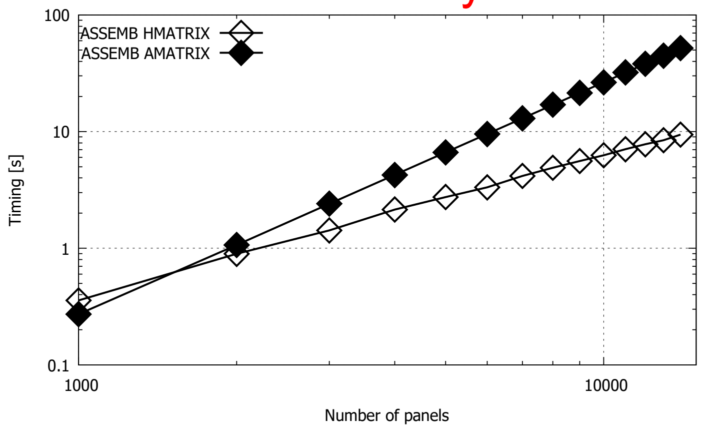
$$\text{Total Time} \sim n\log(n)$$

Results

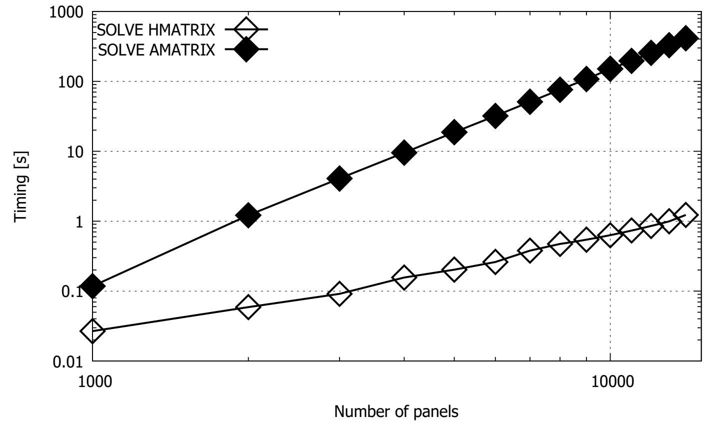
Solution of Laplace Equation using Hierarchical Matrices

Cholesky Decomposition Method

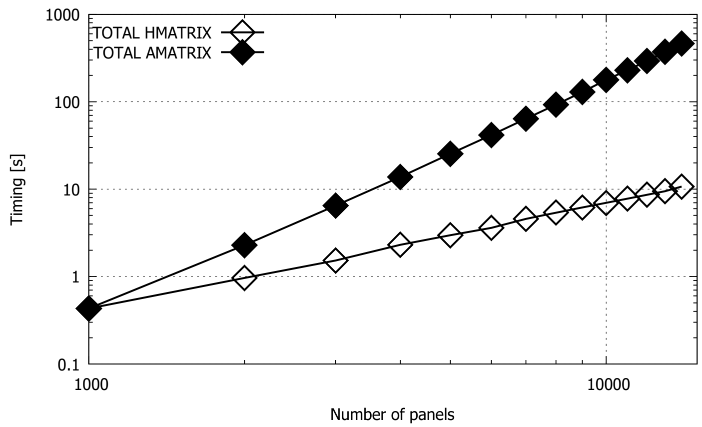
Assembly



Solve



Total



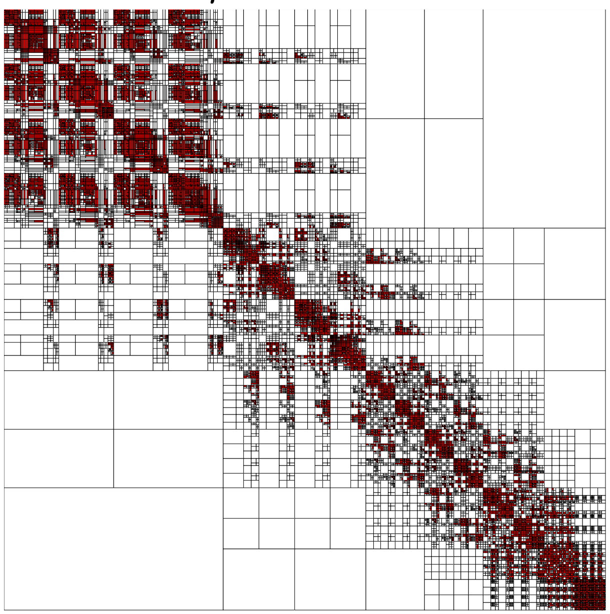
$\text{Total Time} \sim n^3$

$\text{Total Time} \sim n \log(n)$

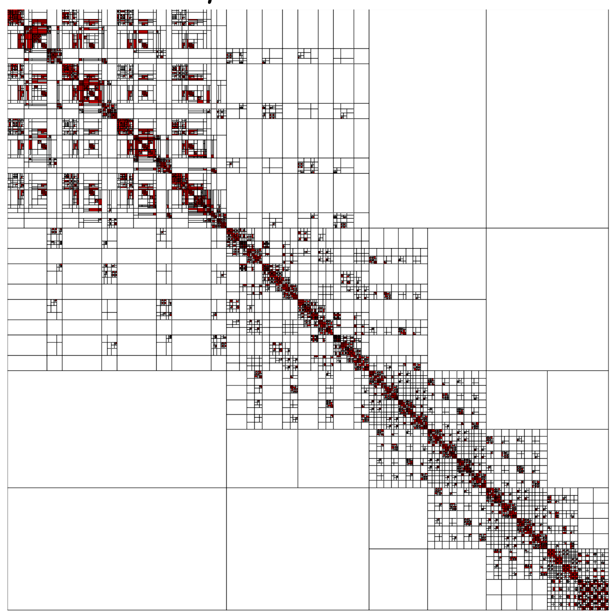
Results

Hierarchical Matrix for the UXO

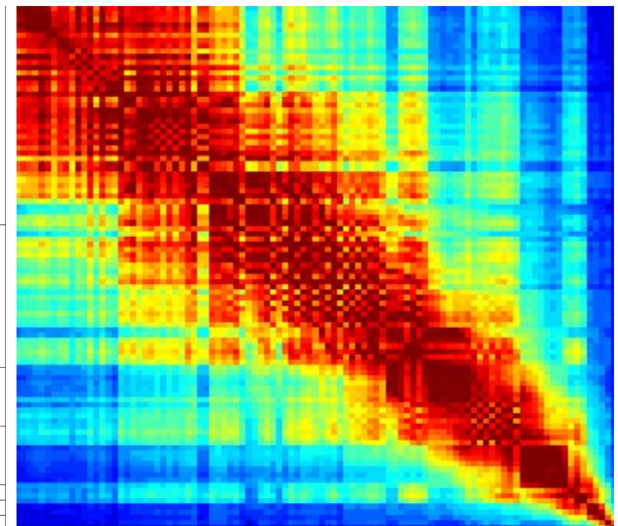
$\eta = 1$



$\eta = 3$

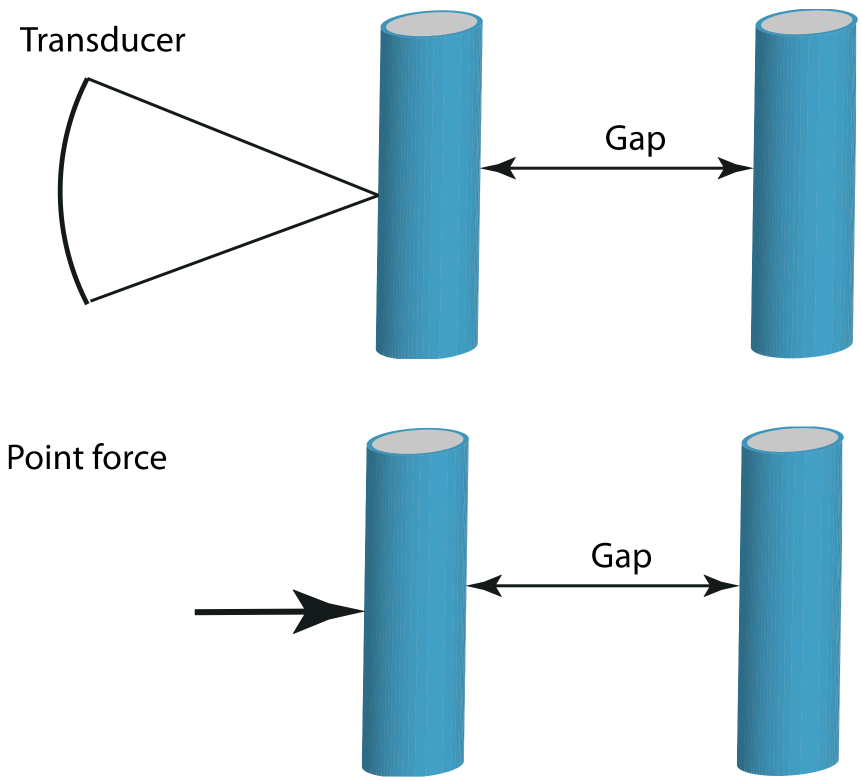


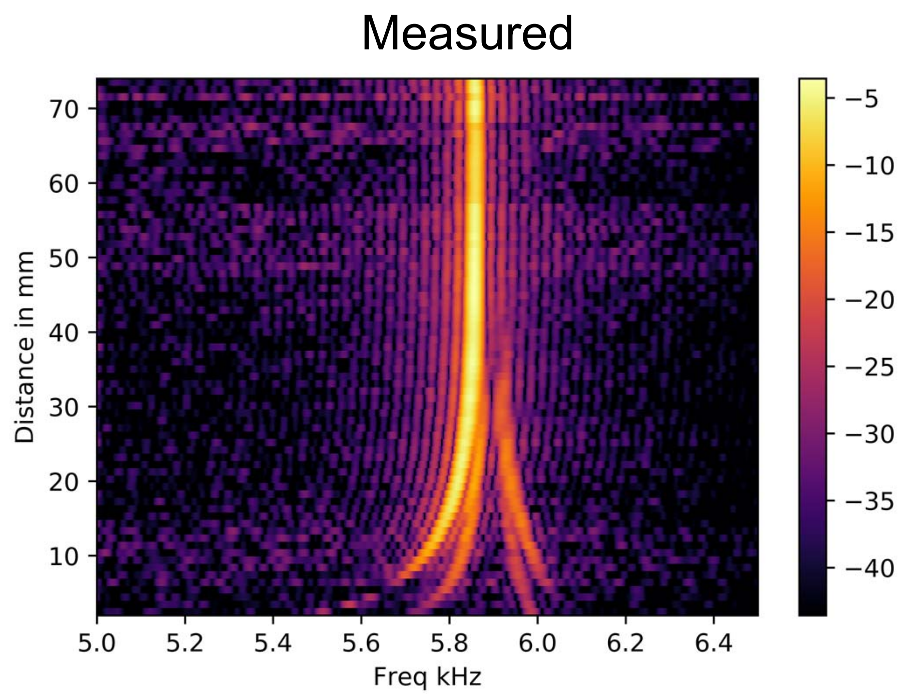
Rank of Sub-Blocks



Results

Scattering from two cylinders

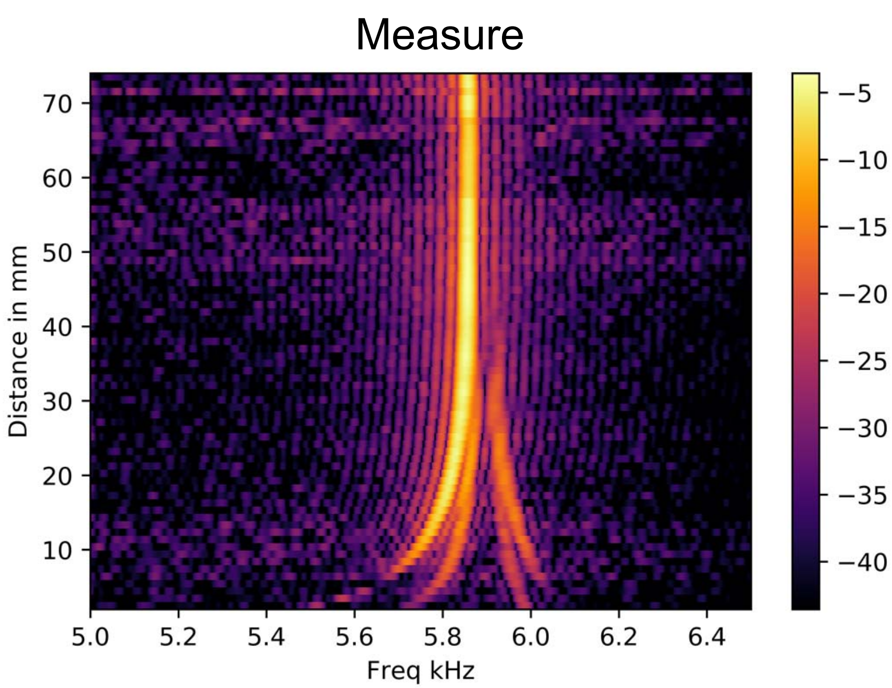


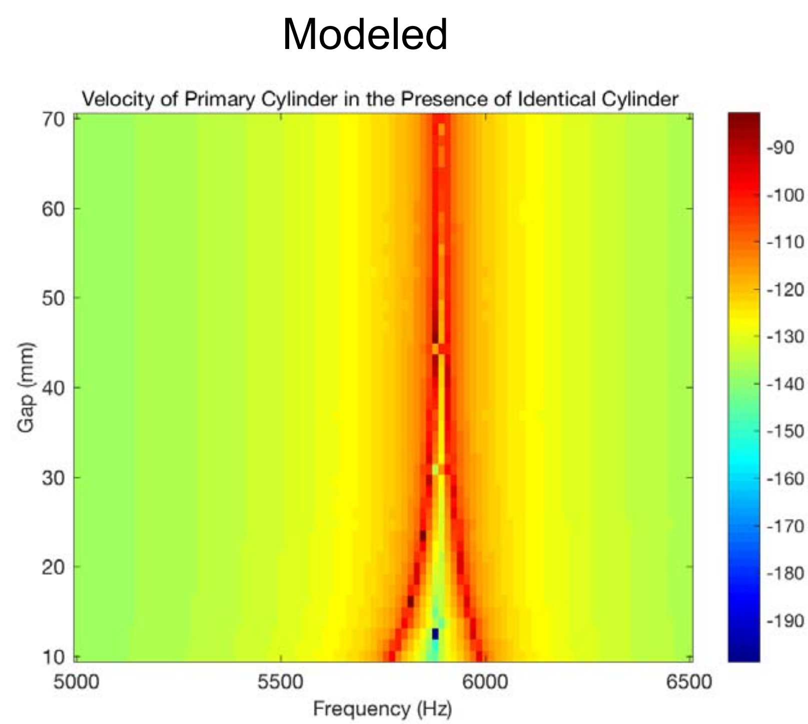


First bending mode is at 5900 Hz

Results

Velocity Vs. frequency and distance





$$\begin{bmatrix} \mathbf{\Omega}_{11} & \mathbf{\Omega}_{12} \\ \mathbf{\Omega}_{21} & \mathbf{\Omega}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{p}_{inc_1} \\ \mathbf{p}_{inc_2} \end{bmatrix}$$

Transition Plan

- The technology developed under this program has direct application to ASW and MCM. It has been validated using other models and experimental data

- Its evolving improved speed makes it ideal for use in training classifiers in automatic target recognition (ATR) efforts

- The use of multi-target models will prove useful not only in studying target scattering in complex environments in the presence of clutter, but in manipulating/analyzing collected data and planning future experiments in such environments

- With the full implementation of H-Matrices, we will be well-equipped to model experiments involving large number of sources/receivers, operating at high ping rates in high density target/clutter fields

- To advertise this technology, we plan to publish our results in refereed journals and talk about them in national and international conferences.

Publications

Ahmad T. Abawi, Petr Krysl "Coupled finite element/boundary element formulation for scattering from axially-symmetric objects in three dimensions", The Journal of Acoustical Society of America, 142, 3637, (2017).

Ahmad T. Abawi, Petr Krysl, Aubrey España, Steve Kargl, Kevin Williams and Dan Plotnick, " Modeling acoustic response of elastic targets in a layered medium using the coupled finite element/boundary element method" , The Journal of Acoustical Society of America, 140(4), 2968, (2016).

BACKUP MATERIAL

These charts are required, but will only be briefed if questions arise.

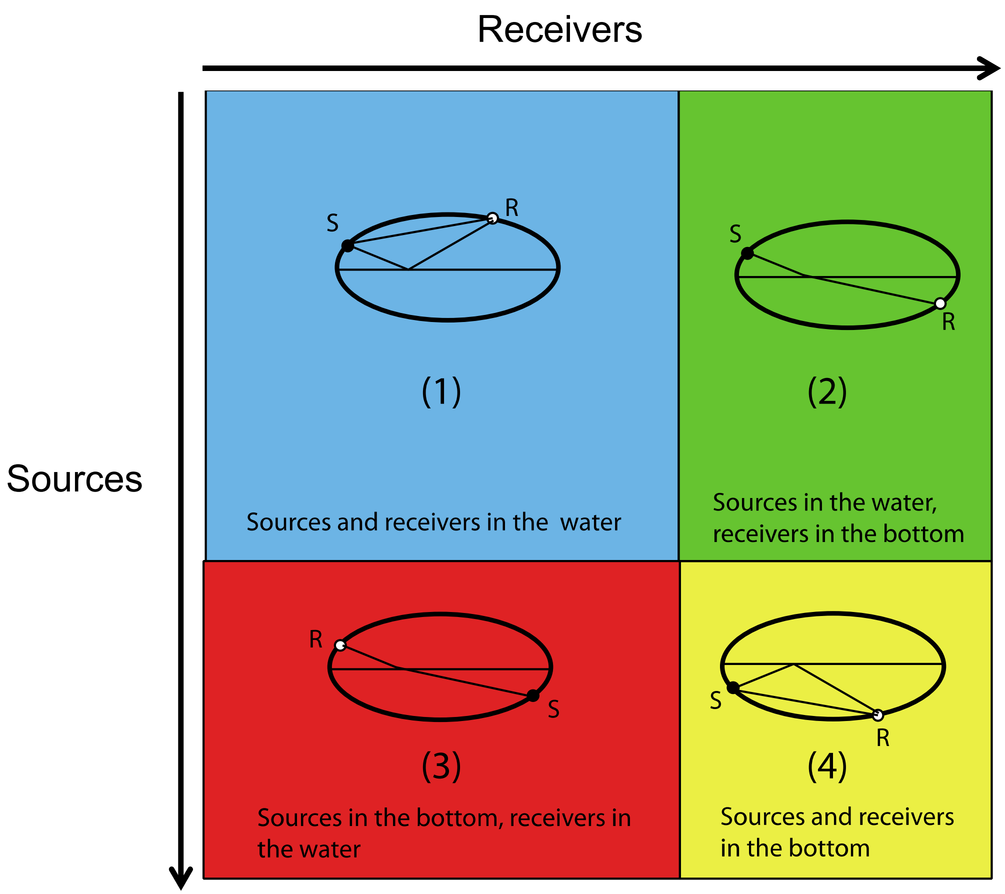
Problem Statement

- This technology is developing computational tools to model the response of UXOs to a sonar signal in the complex environments where they reside to
  - ♦ Understand and interpret experimental data
  - ♦ Simulate realistic UXO-hunting scenarios
  - ♦ Design experiments
- The current modeling approach is based on the coupled finite/boundary element method

Technical Approach

Background

When sources and receivers are in two media (partially-buried targets), decompose the interaction matrix into 4 sub-matrices



Matrices 1 and 4 are computed using interpolation

Matrix 2 is computed directly

Matrix 3 is computed using reciprocity

$$g_3(k_r, z, z_s) = \frac{\rho_1}{\rho_2} g_2(k_r, z, z_s)^T,$$

$$\frac{dg_3(k_r, z, z_s)}{dx} = -\frac{\rho_1}{\rho_2}\left(\frac{dg_2(k_r, z, z_s)}{dx}\right)^T,$$

$$\frac{dg_3(k_r, z, z_s)}{dy} = -\frac{\rho_1}{\rho_2}\left(\frac{dg_2(k_r, z, z_s)}{dy}\right)^T,$$

$$\frac{dg_3(k_r, z, z_s)}{dz} = -ik_{z_2}\frac{\rho_1}{\rho_2} g_2(k_r, z, z_s)^T.$$

Admissibility condition:

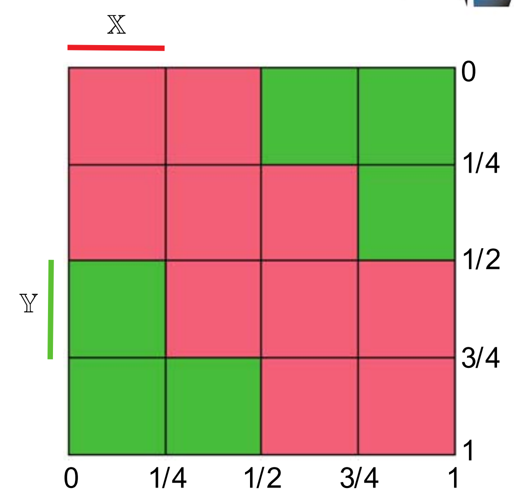
$$\max\{\operatorname{diam}(\mathbb{X}), \operatorname{diam}(\mathbb{Y})\} \leq \eta \operatorname{dist}(\mathbb{X}, \mathbb{Y})$$

$$\operatorname{diam}(\mathbb{X}) = \max|x - y|,\ \ x, y \in \mathbb{X},$$

$$\operatorname{diam}(\mathbb{Y}) = \max|x - y|,\ \ x, y \in \mathbb{Y},$$

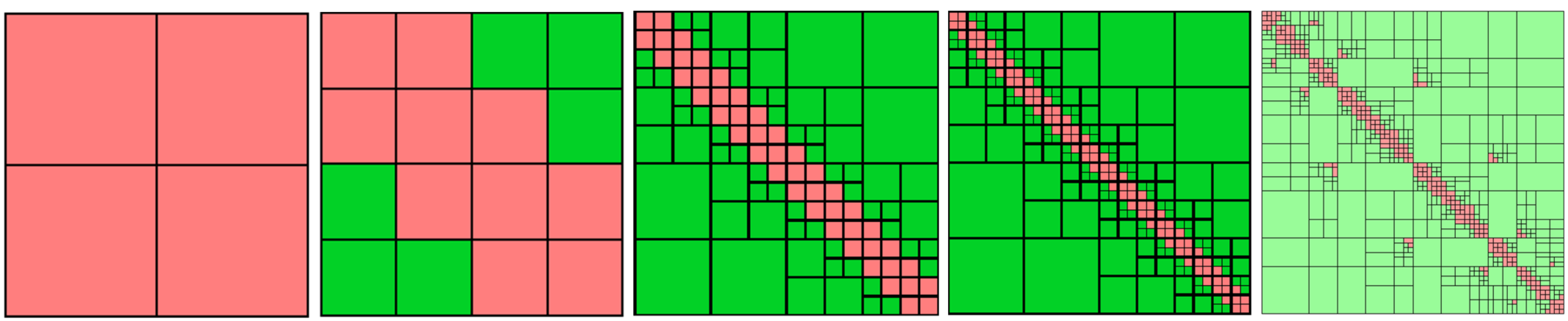
$$\operatorname{dist}(\mathbb{X}, \mathbb{Y}) = \inf|x - y|,\ \ x \in \mathbb{X},\ \ y \in \mathbb{Y}.$$

η is an adjustable parameter



Low-rank representation: A rank k representation of $X \in \mathbb{R}^{n \times m},\ k \in \mathbb{N}$ is a factorization of the form $X = AB^T$ with matrices $A \in \mathbb{R}^{n \times k},\ B \in \mathbb{R}^{m \times k}$

Continuing the process results in hierarchical matrices, reducing both matrix operations and assembly from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log N)$, where N is the original size of the matrix.



Technical Approach

The multipole solution in two half-spaces

Solve Helmholtz equation in cylindrical coordinates for a source located at (r_s, z_s, φ_s)

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}+\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}+k^2\right]\psi\left(r,z,\varphi\right)=S_w\frac{\delta(r-r_s)\delta(z-z_s)\delta(\varphi-\varphi_s)}{2\pi r}$$

Let $\psi\left(r,z,\varphi\right)=\sum_{n=-\infty}^{\infty}\psi_n(r,z)e^{in\varphi},$

Then for each n, we get

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}-\frac{n^2}{r^2}+k^2\right]\psi_n\left(r,z\right)=S_w\frac{\delta(r-r_s)\delta(z-z_s)}{2\pi r}e^{-in\varphi_s}$$

Applying the Hankel transform

$$\psi_n\left(r,z\right)=\int_0^{\infty}\psi_n(k_r,z)J_n(k_r r)k_r dk_r,$$

Technical Approach

The multipole solution in two half-spaces

and expressing $\frac{\delta(r-r_s)}{r} = \int_0^\infty J_n(k_r r) J_n(k_r r_s) k_r dk_r$, gives the depth-separated,

spectral wave equation

$$\left[\frac{d^2}{dz^2} + \left(k^2 - k_r^2\right)\right] \psi_n(k_r, z) = S_w \frac{\delta(z-z_s)}{2\pi} e^{-in\varphi_s} J_n(k_r r_s),$$

which has a solution

$$\psi\left(r_r, z_r, \varphi_r; r_s, z_s, \varphi_s\right) = \frac{iS_w}{4\pi} \sum_{n=0}^{\infty} \epsilon_n \int_0^\infty \frac{e^{ik_z|z_r - z_s|}}{k_z} J_n(k_r r_r) J_n(k_r r_s) \cos[n(\varphi_r - \varphi_s)] k_r dk_r$$

This is the multipole representation of the free space Green's function in cylindrical coordinates, which can be expressed as an outer product.

This equation is the same as Eq. (7.3.15) of Morse and Ingard

Results

Exact scattering from two targets

The boundary element formulation is governed by

$$\mathbf{A}\mathbf{p} = \mathbf{B}\mathbf{v}_n + \mathbf{p}_{inc}, \quad \mathbf{f} = -\mathbf{L}\mathbf{p}, \quad \mathbf{v}_n = -i\omega\mathbf{D}^{-1}\mathbf{L}^T\mathbf{u},$$

and displacement and force are related by $\left(-\omega^2\mathbf{M} + \mathbf{K}\right)\mathbf{u} = \mathbf{f}$.

Application of the first equation to two targets gives

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v}_{n_1} \\ \mathbf{v}_{n_2} \end{bmatrix} + \begin{bmatrix} \mathbf{p}_{inc_1} \\ \mathbf{p}_{inc_2} \end{bmatrix}$$

Where

$$\mathbf{A}_{11} \in \mathbb{C}^{n\times n}, \ \mathbf{A}_{12} \in \mathbb{C}^{n\times m}, \ \mathbf{A}_{21} \in \mathbb{C}^{m\times n}, \ \mathbf{A}_{22} \in \mathbb{C}^{m\times m}, \ \mathbf{p}_1 \in \mathbb{C}^{n}, \ \mathbf{p}_2 \in \mathbb{C}^{m}.$$

Using the above equations,

$$\mathbf{v}_{n_\ell} = i\omega\mathbf{\Gamma}_\ell\mathbf{p}_\ell, \ \mathbf{\Gamma}_\ell = \mathbf{D}_\ell^{-1}\mathbf{L}_\ell^T\left(-\omega^2\mathbf{M}_\ell + \mathbf{K}_\ell\right)^{-1}\mathbf{L}_\ell, \ \ell = 1, 2.$$

Substituting these in the above matrix equation and solving for the pressure gives

Results

Exact scattering from two targets

$$\begin{bmatrix} \mathbf{\Omega}_{11} & \mathbf{\Omega}_{12} \\ \mathbf{\Omega}_{21} & \mathbf{\Omega}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{p}_{inc_1} \\ \mathbf{p}_{inc_2} \end{bmatrix},$$

where

$$\mathbf{\Omega}_{11} = \mathbf{A}_{11} - i\omega \mathbf{B}_{11} \mathbf{\Gamma}_1,$$

$$\mathbf{\Omega}_{12} = \mathbf{A}_{12} - i\omega \mathbf{B}_{12} \mathbf{\Gamma}_2,$$

$$\mathbf{\Omega}_{21} = \mathbf{A}_{21} - i\omega \mathbf{B}_{21} \mathbf{\Gamma}_1,$$

$$\mathbf{\Omega}_{22} = \mathbf{A}_{22} - i\omega \mathbf{B}_{22} \mathbf{\Gamma}_2.$$